Substituting from the original equation and (3) into (2), we have

$$x\frac{dy}{dx} + \tfrac{1}{2}y = \frac{y}{x}. \tag{4}$$

From this we have

$$\log y = -1/x - \tfrac{1}{2}\log x + \log C,$$

or

$$y = Ce^{-(1/x)}x^{-\frac{1}{2}}.$$

## II. Solution by the Proposer.

According to Professor Kelland (*Trans. Royal Society of Edinburgh*, Vols. XIV and XVI) the general differential operator may be defined as follows:

$$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu},$$

for all values of $n$ and $\mu$. We have $\Gamma(n+1) = n\Gamma(n)$.

Let us assume $y = A_0 + A_1x^{-\frac{1}{2}} + A_2x^{-1} + A_3x^{-\frac{3}{2}} + \cdots$.

Then

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = (-1)^{\frac{1}{2}}\left\{\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1x^{-1} + \frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2x^{-\frac{3}{2}} + \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3x^{-2} + \cdots\right\}$$

and

$$\frac{y}{x} = A_0x^{-1} + A_1x^{-\frac{3}{2}} + A_2x^{-2} + A_3x^{-\frac{5}{2}} + \cdots.$$

Hence,

$$i\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}A_1 = A_0 \text{ and } A_1 = \frac{\sqrt{\pi}}{i}A_0 = -i\sqrt{\pi}A_0.$$

For, since $\Gamma(p)\Gamma(1-p) = \pi/\sin p\pi$ when $p$ is a fraction less than one, then $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Also, $\Gamma(1) = 1$.

Furthermore,

$$i\frac{\Gamma(\frac{3}{2})}{\Gamma(1)}A_2 = A_1 = -i\sqrt{\pi}A_0, \text{ or } A_2 = -2A_0,$$

since $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$; also

$$i\frac{\Gamma(2)}{\Gamma(\frac{3}{2})}A_3 = A_2 = -2A_0, \text{ or } A_3 = -\frac{\sqrt{\pi}}{i}A_0 = i\sqrt{\pi}A_0.$$

Finally, $y = A_0(1 - i\sqrt{\pi}x^{-\frac{1}{2}} - 2x^{-1} + i\sqrt{\pi}x^{-\frac{3}{2}} + \cdots)$.

*Note.*—This problem, which is very similar to one solved by Professor Kelland, was submitted because it was thought that it might prove of interest to the readers of the Monthly. The proposer would be glad to see some discussion of the subject of general differentiation.

The above solution may be considered as a reply to the following question:

**360 (Calculus.) Proposed by ELMER SCHUYLER, Brooklyn, New York.**

What interpretation must be given to

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} \text{ so that } \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right) = \frac{dy}{dx}?$$

**435 (Calculus). Proposed by B. F. FINKEL, Drury College.**

Show that

$$\int_0^\infty e^{-x^2-a^2/x^2}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation, rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, page 106–107.

SOLUTION BY OTTO J. RAMLER, Catholic University of America.

We have the known definite integral

$$\int_{-\infty}^{\infty} e^{-z^2}dz = \sqrt{\pi}.$$

Now let $z = (x - a/x)$. Then

$$dz = (1 + a/x^2)dx,$$

and the integral becomes

$$\int_0^{\infty} e^{-(x-a/x)^2}(1 + a/x^2)dx = \sqrt{\pi},$$

or

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + a\int_0^{\infty} e^{-(x-a/x)^2}\frac{dx}{x^2} = \sqrt{\pi}.$$

In the second integral, let $x = a/y$. Then $dx = -a/y^2dy$, and the second integral becomes

$$\int_0^{\infty} e^{-(a/y-y)^2}dy \equiv \int_0^{\infty} e^{-(y-a/y)^2}dy.$$

Hence,

$$\int_0^{\infty} e^{-(x-a/x)^2}dx + \int_0^{\infty} e^{-(y-a/y)^2}dy = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-(x-a/x)^2}dx = \sqrt{\pi}$$

or

$$2\int_0^{\infty} e^{-x^2-a^2/x^2+2a}dx = 2e^{2a}\int_0^{\infty} e^{-x^2-a^2/x^2}dx = \sqrt{\pi}.$$

Whence,

$$\int_0^{\infty} e^{-(x^2-a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}.$$

Solved similarly by O. S. ADAMS.

**339 (Mechanics). Proposed by C. N. SCHMALL, New York City.**

A roll of cloth of very small uniform thickness $a$ is coiled up tightly in the form of a circular cylinder of diameter $d$ and is laid horizontally across a perfectly rough incline so that its axis is parallel to the intersection of the plane with the horizontal. It is then allowed to unroll (without slipping) down the plane. Neglecting the motion of its center of gravity in the direction perpendicular to the plane, show that it will unroll entirely in the time

$$T = \frac{\pi}{4}\sqrt{\frac{6d^2}{ag\sin\phi}},$$

where $\phi$ is the inclination of the plane to the horizontal plane, and $g$ is the acceleration of gravity.

SOLUTION BY PAUL CAPRON, U. S. Naval Academy.

After $t$ seconds, let the radius of the cylinder be $r$; let $s$ be the distance its center of gravity has moved parallel to the plane; $\theta$, the angle through which it has turned; $W = mg = 2\rho g r^2$ its weight; and let the reaction against it at its point of application with the plane be composed of $R$ lbs. perpendicular to the plane and $S$ lbs. upward along the plane. The moment of inertia is then $I = \rho r^4$. Since $a$ is very small, assume

$$r = \frac{d}{2} - \frac{a\theta}{2\pi}, \qquad r d\theta = ds.$$

Using the dot and double dot to indicate first and second derivatives with respect to the time, we have the equations for motion of the center of gravity along and perpendicular to the plane and of the cylinder about its geometric axis:

$$\frac{d}{dt}(m\dot{s}) = W\sin\phi - S, \qquad \frac{d}{dt}(m\dot{r}) = R - W\cos\phi, \qquad \text{and} \qquad \frac{d}{dt}(I\dot{\theta}) = rS.$$